Turbulent flat plate:

$$N=0.8 \quad V \leq 3962 \text{ m/s} \quad M=3.37$$

$$C_2 = 3.35(10^{-8})(\cos\phi)^{1.78}(\sin\phi)^{1.6} x_T^{-1/5} \times (T_w/556)^{-1/4}(1-1.11 g_w)$$

$$N=0.8 \quad V > 3962 \text{ m/s} \quad M=3.7$$

$$C_2 = 2.20(10^{-9})(\cos\phi)^{2.08}(\sin\phi)^{1.6} x_T^{-1/5}(1-1.11 g_w)$$

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Evaluation of Emission Integrals for the Radiative Transport Equation

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Introduction

ONE difficulty associated with discrete ordinates solutions to the radiative transport equation[1] is the accurate and computationally efficient evaluation of the emission integrals that appear in the particular solutions.[2-4] Experience has shown that most of the computational effort required in the determination of the radiative heat fluxes in participating media is due to time spent evaluating these emission integrals.[5] The purpose of this Note is to present and compare two efficient techniques for evaluating the emission integrals for situations involving arbitrary temperature profiles.

Radiative Transport Equation

The general one-dimensional axisymmetric integrodifferential equation of radiative transfer in a plane parallel medium that absorbs, emits, and anisotropically scatters is given by[4]

$$\frac{dI(\tau,\mu)}{d\tau} = -\frac{I(\tau,\mu)}{\mu} + \frac{W}{2\mu}\int_{-1}^{1} I(\tau,\mu')\Phi(\mu,\mu')d\mu' + \frac{(1-W)n^2 I_b(\tau)}{\mu} \tag{1}$$

Here, τ is the optical depth, μ the cosine of the polar angle θ, W the albedo of scattering, $I(\tau,\mu)$ the radiative intensity at depth τ and in the direction given by μ, n the refactive index, $I_b(\tau)$ Planck's blackbody intensity function, and $\Phi(\mu,\mu')$ the scattering phase function. The phase function can be represented by an N-term series of Legendre polynomials as

$$\Phi(\mu,\mu') = \sum_{\ell=0}^{N} A_\ell P_\ell(\mu) P_\ell(\mu') \tag{2}$$

where the coefficients A_ℓ can be determined either analytically or through the use of experimental data. If scattering within the medium is isotropic rather than anisotropic, the phase function takes on a value of unity.

Replacing the integral term in Eq. (1) with an m-order Gauss-Legendre quadrature yields the discrete ordinate approximation of the radiative transport equation, i.e.,

$$\frac{dI(\tau,\mu_i)}{d\tau} = -\frac{I(\tau,\mu_i)}{\mu_i} + \frac{W}{2\mu_i}\sum_{j=1}^{m} a_j I(\tau,\mu_j)\sum_{\ell=0}^{N} A_\ell P_\ell(\mu_i)P_\ell(\mu_j) + \frac{(1-W)n^2 I_b(\tau)}{\mu_i} \qquad i=1,\ldots,m \tag{3}$$

where the μ_i and μ_j are the quadrature points, and the a_j are the quadrature weights. Equation (3) represents a system of first-order, linear, inhomogeneous, ordinary differential equations, and solutions to this set of equations have been obtained for situations involving various levels of complexity.[1-4] The set of general solutions to Eq. (3) is given by[4]

$$\begin{aligned} I(\tau,\mu_i) = & \sum_{j=1}^{m/2}(1-\lambda_j\mu_j)\left[\frac{C_j e^{\lambda_j\tau} Y(\mu_i,\lambda_j)}{1+\mu_i\lambda_j} + \frac{C_{m+1-j} e^{-\lambda_j\tau} Y(\mu_i,-\lambda_j)}{1-\mu_i\lambda_j}\right] \\ & + (1-W)n^2 \sum_{j=1}^{m/2}\frac{K_{m+1-j} Y(\mu_i,-\lambda_j)}{1-\mu_i\lambda_j}\int_0^{\tau} I_b(t) e^{-\lambda_j(\tau-t)}dt \\ & - (1-W)n^2 \sum_{j=1}^{m/2}\frac{K_j Y(\mu_i,\lambda_j)}{1+\mu_i\lambda_j}\int_{\tau}^{\tau_0} I_b(t) e^{-\lambda_j(t-\tau)}dt \qquad i=1,\ldots,m \end{aligned} \tag{4}$$

where τ_0 is the optical thickness of the medium, C_j the m constants of integration that must satisfy a given set of boundary conditions, and λ_j the $m/2$ positive eigenvalues for the system of equations described by Eq. (3). The Y functions are related to the phase function, and K_j are a set of constants that must satisfy the system of equations given by

$$\sum_{j=1}^{m/2} K_j\left[\frac{Y(\mu_i,\lambda_j)}{1+\mu_i\lambda_j} - \frac{Y(\mu_i,-\lambda_j)}{1-\mu_i\lambda_j}\right] = \frac{1}{\mu_i}, \qquad i=1,\ldots,m/2 \tag{5}$$

$$K_{m+1-j} = -K_j, \qquad j=1,\ldots,m/2 \tag{6}$$

The net radiative heat flux at depth τ can be obtained by integrating the intensity as follows:

$$Q_R(\tau) = 2\pi\int_{-1}^{1} I(\tau,\mu)\mu d\mu \cong 2\pi\sum_{i=1}^{m} I(\tau,\mu_i)\mu_i a_i \tag{7}$$

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Substituting Eq. (4) into Eq. (7) and doing some manipulation gives

$$Q_R(\tau) = -\frac{4\pi(1-W)}{W}\sum_{j=1}^{m/2}\frac{(1-\lambda_j\mu_j)}{\lambda_j}[C_j e^{\lambda_j\tau} - C_{m+1-j}e^{-\lambda_j\tau}]$$
$$-\frac{4\pi(1-W)^2 n^2}{W}\sum_{j=1}^{m/2}\frac{K_j}{\lambda_j}\left[\int_0^\tau I_b(t)e^{-\lambda_j(\tau-t)}\mathrm{d}t\right.$$
$$\left.-\int_\tau^{\tau_0} I_b(t)e^{-\lambda_j(t-\tau)}\mathrm{d}t\right] \tag{8}$$

Results obtained during a recent study[5] indicate that the accuracy of the net radiative flux, as computed from Eq. (8), is highly dependent on the accuracy to which the integral terms can be evaluated. Therefore, the remaining discussion is devoted to techniques for evaluating the emission integrals. In order to simplify our discussion, the medium is assumed to be gray and isotropically scattering. Sample calculations are included for selected linear and quadratic temperature profiles in order to quantify the results.

Integration by Parts

The emission integrals under consideration here are given by

$$I_1(\tau) = \int_0^\tau i_b(t)\exp[-\lambda_j(\tau-t)]\mathrm{d}t \tag{9}$$

$$I_2(\tau) = \int_\tau^{\tau_0} i_b(t)\exp[-\lambda_j(t-\tau)]\mathrm{d}t \tag{10}$$

$$i_b(t) = I_b(t)/I_0 \tag{11}$$

where I_0 is some arbitrary radiative intensity used for normalization. Now, a commonly used approach in evaluating integrals of this type is integration by parts. Performing the integrations for Eqs. (9) and (10) yields

$$I_1(\tau) = \frac{i_b(\tau)}{\lambda_j} - \frac{i_b'(\tau)}{\lambda_j^2} + \frac{i_b''(\tau)}{\lambda_j^3} - \ldots + \frac{(-1)^{\ell-1}i_b^{[\ell-1]}(\tau)}{\lambda_j^\ell}$$
$$-\left[\frac{i_b(0)}{\lambda_j} - \frac{i_b'(0)}{\lambda_j^2} + \frac{i_b''(0)}{\lambda_j^3} - \ldots + \frac{(-1)^{\ell-1}i_b^{[\ell-1]}(0)}{\lambda_j^\ell}\right]$$
$$\times\exp(-\lambda_j\tau) - \frac{(-1)^{\ell-1}}{\lambda_j^\ell}\int_0^\tau i_b^{[\ell]}(t)\exp[-\lambda_j(\tau-t)]\mathrm{d}t \tag{12}$$

and

$$I_2(\tau) = \frac{i_b(\tau)}{\lambda_j} + \frac{i_b'(\tau)}{\lambda_j^2} + \frac{i_b''(\tau)}{\lambda_j^3} + \ldots + \frac{i_b^{[\ell-1]}(\tau)}{\lambda_j^\ell}$$
$$-\left[\frac{i_b(\tau_0)}{\lambda_j} + \frac{i_b'(\tau_0)}{\lambda_j^2} + \frac{i_b''(\tau_0)}{\lambda_j^3} + \ldots + \frac{i_b^{[\ell-1]}(\tau_0)}{\lambda_j^\ell}\right]$$
$$\times\exp[-\lambda_j(\tau_0-\tau)] + \frac{1}{\lambda_j^\ell}\int_\tau^{\tau_0} i_b^{[\ell]}(t)\exp[-\lambda_j(\tau-t)]\mathrm{d}t \tag{13}$$

Upon noting that $i_b(\tau)$ is, in reality, an implicit function of τ, i.e., $i_b(\tau) = i_b(T(\tau))$, the derivative terms in Eqs. (12) and (13) can be written as

$$i_b' = \frac{\mathrm{d}i_b(T)}{\mathrm{d}\tau} = T'\frac{\mathrm{d}i_b(T)}{\mathrm{d}T} \tag{14}$$

$$i_b'' = \frac{\mathrm{d}^2 i_b(T)}{\mathrm{d}\tau^2} = (T')^2\frac{\mathrm{d}^2 i_b(T)}{\mathrm{d}T^2} + T''\frac{\mathrm{d}i_b(T)}{\mathrm{d}T} \tag{15}$$

and so forth, where $T = T(\tau)$ represents the temperature profile throughout the medium. Evaluation of $I_1(\tau)$ and $I_2(\tau)$ can be accomplished by noting that successive temperature derivatives of $i_b(T(\tau))$ decrease rapidly in magnitude. Thus, only a finite number of terms need be retained in Eqs. (12) and (13), provided that the temperature profile is well behaved. For the case of a gray medium, the emitted radiative intensity behaves according to the Stefan-Boltzmann law ($I_b(T) = \sigma T^4/\pi$); therefore, derivatives of fifth or higher order with respect to temperature make no contribution to the integrals. Experience has shown that retention of the first derivative terms is usually sufficient in cases where temperature gradients are small. If the temperature profiles within the medium are arbitrary (such as when solving coupled conduction and radiation problems), the spatial derivatives of $i_b(\tau)$ must be determined numerically.

Numerical Approach

For cases involving large temperature gradients, programming considerations render it impractical to use the integration by the parts approach cited earlier. This difficulty can be overcome by dividing the medium into a finite number of elements (grid fineness is dictated by both the temperature range and the nonlinearity of the temperature profiles) and then assuming that the spatial variation of the emitted radiative intensity within the medium is linear between two adjacent nodes. The emission integrals are then evaluated by treating them as the sum of the integrals between each pair of nodes up to the point of interest as follows:

$$I_1(\tau) = \int_0^\tau i_b(t)e^{-\lambda_j(\tau-t)}\mathrm{d}t$$
$$\cong \sum_{i=1}^{N-1}\int_{\tau_i}^{\tau_{i+1}}(S_i t + B_i)\exp[-\lambda_j(\tau_N - t)]\mathrm{d}t \tag{16}$$

$$I_2(\tau) = \int_\tau^{\tau_0} i_b(t)e^{-\lambda_j(t-\tau)}\mathrm{d}t$$
$$\cong \sum_{i=N}^{NP-1}\int_{\tau_i}^{\tau_{i+1}}(S_i t + B_i)\exp[-\lambda_j(t - \tau_N)]\mathrm{d}t \tag{17}$$

where N is the node number of the spatial location at which the integrals are desired, NP is the top boundary node number, and S_i and B_i are given by

$$S_i = \frac{i_b(\tau_{i+1}) - i_b(\tau_i)}{\tau_{i+1} - \tau_i}, \qquad i \leq NP - 1 \tag{18}$$

$$B_i = i_b(\tau_i) - S_i\tau_i, \qquad i \leq NP - 1 \tag{19}$$

Integrating by parts and performing some algebra yields

$$I_1(\tau_N) \cong \sum_{i=1}^{N-1}\left\{\left[\frac{i_b(\tau_{i+1})}{\lambda_j} - \frac{S_i}{\lambda_j^2}\right]\exp[-\lambda_j(\tau_N - \tau_{i+1})]\right.$$
$$\left.-\left[\frac{i_b(\tau_i)}{\lambda_j} - \frac{S_i}{\lambda_j^2}\right]\exp[-\lambda_j(\tau_N - \tau_i)]\right\} \tag{20}$$

$$I_2(\tau_N) \cong \sum_{i=N}^{NP-1}\left\{\left[\frac{i_b(\tau_i)}{\lambda_j} + \frac{S_i}{\lambda_j^2}\right]\exp[-\lambda_j(\tau_i - \tau_N)]\right.$$
$$\left.-\left[\frac{i_b(\tau_{i+1})}{\lambda_j} + \frac{S_i}{\lambda_j^2}\right]\exp[-\lambda_j(\tau_{i+1} - \tau_N)]\right\} \tag{21}$$

where $I_1(\tau_1) = 0$ and $I_2(\tau_{NP}) = 0$. Now, Eqs. (20) and (21) may be placed in a more convenient form for further discus-

Table 1 Comparisons for I_1 at $\tau_0 = 10$

Linear temperature profile: $T(\tau) = 5(6.0\tau + 539.69)/9$

j	λ_j	Exact	Parts	Error, %	Numerical	Error, %
1	1.0033147	0.958077987	0.956912138	−0.121686179	0.958078964	0.000102060
2	1.0461751	0.920309082	0.919279880	−0.111832210	0.920310020	0.000101979
3	1.1373957	0.849061507	0.848259365	−0.094473946	0.849062372	0.000101825
4	1.2995679	0.746327752	0.745788805	−0.072213095	0.746328510	0.000101602
5	1.5852462	0.615189108	0.614891353	−0.048400447	0.615189731	0.000101314
6	2.1350000	0.459726478	0.459604195	−0.026599080	0.459726942	0.000100958
7	3.4687170	0.284993463	0.284964847	−0.010041146	0.284993749	0.000100473
8	10.275242	0.096943360	0.096942255	−0.001139961	0.096943455	0.000098563

Quadratic temperature profile: $T(\tau) = 5(-0.75\tau^2 + 13.5\tau + 539.69)/9$

j	λ_j	Exact	Parts	Error, %	Numerical	Error, %
1	1.0033147	0.996775093	1.006609577	0.986630120	0.996766845	−0.000827516
2	1.0461751	0.956306181	0.964988143	0.907864254	0.956298264	−0.000827887
3	1.1373957	0.880163875	0.886929435	0.768670508	0.880156583	−0.000828497
4	1.2995679	0.770866878	0.775409635	0.589305008	0.770860486	−0.000829160
5	1.5852462	0.632292273	0.634798134	0.396313781	0.632287027	−0.000829612
6	2.1350000	0.469552909	0.470579048	0.218535230	0.469549014	−0.000829498
7	3.4687170	0.288883545	0.289122581	0.082744583	0.288881153	−0.000828027
8	10.275242	0.097406900	0.097416071	0.009415665	0.097406107	−0.000814320

sion as follows:

$$I_1(\tau_N) \cong \sum_{i=1}^{N-1} \{D_i \exp[-\lambda_j(\tau_N - \tau_{i+1})] - E_i \exp[-\lambda_j(\tau_N - \tau_i)]\} \tag{22}$$

$$I_2(\tau_N) \cong \sum_{i=N}^{NP-1} \{F_i \exp[-\lambda_j(\tau_i - \tau_N)] - G_i \exp[-\lambda_j(\tau_{i+1} - \tau_N)]\} \tag{23}$$

where D_i, E_i, F_i, and G_i are the ith coefficients of the exponential terms. It is now noted that, in the present form, the evaluation of the emission integrals can be computationally tedious. However, multiplying both sides of Eq. (22) by $\exp[-\lambda_j(\tau_{N+1} - \tau_N)]$ and multiplying both sides of Eq. (23) by $\exp[-\lambda_j(\tau_N - \tau_{N+1})]$ leads to the following set of recursion relations:

$$I_1(\tau_M) = [I_1(\tau_N) - E_{M-1}]\exp[-\lambda_j(\tau_M - \tau_N)] + D_{M-1}, \quad 2 \leq M \leq NP \tag{24}$$

$$I_2(\tau_N) = [I_2(\tau_M) - G_N]\exp[-\lambda_j(\tau_M - \tau_N)] + F_N, \quad 1 \leq M \leq NP - 1 \tag{25}$$

where $M = N + 1$. These recursion relations can be used to obtain the integral of the Mth term from the integral of the Nth term in the case of I_1 and the integral of the Nth term from the integral of the Mth term in the case of I_2. The evaluation procedure should be started at $N = 2$ for I_1 and at $N = NP - 1$ for I_2 in order to take advantage of the nodes for which the respective integrals have a value of zero.

Comparison of Methods

Exact representations of the emission integrals can be obtained through the use of integration by parts for the case of a gray medium in situations where the temperature profiles can be represented as polynomials of order N. Table 1 contains comparisons of approximate values of I_1 for $\tau_0 = 10$ with exact values of I_1 for both linear and parabolic temperature profiles. The lower limit of integration is 0 and the upper limit is τ_0. The approximate values of I_1 are computed from Eq. (12) (parts), keeping only the first derivative terms, and from Eq. (20) (numerical), using 101 nodes. The medium was assumed to be gray and isotropically scattering with an albedo of 0.25, and the eigenvalues λ_j were determined for a 16-point single Gaussian quadrature. The lower and upper boundary temperatures were 299.83 K and 333.16 K, respectively, and the intensity functions were normalized with respect to the upper boundary temperature. The results indicate reasonable agreement for Eq. (12) and excellent agreement for Eq. (20). It should also be noted that reducing the number of nodes used in Eq. (20) by a factor of 2 has little effect on the accuracy. As would be expected, the percentage error for the quadratic temperature profiles increases by about a factor of 8 over that of the linear temperature profiles. However, the magnitude of the error in the results from Eq. (12) decreases significantly with increasing λ_j; whereas, the magnitude of the eigenvalue has very little effect on the error in Eq. (20). Also, for geometrically similar temperature profiles with the same boundary temperatures, the error in I_1 for Eq. (12) decreases rapidly with increasing τ_0, while errors remain at about the same level in Eq. (20). These observations hold for a fairly large range of albedos, and the same trends are observed in similar calculations for I_2.

The use of either of the two methods discussed here can lead to significant savings in the amount of computational time required to evaluate the emission integrals of Eqs. (9) and (10) over that required by standard numerical integration techniques, such as Simpson's rule or the trapezoidal rule. In most cases, however, the extreme accuracy and ease of programming make the numerical procedure the method of choice.

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